MODULE 4

Vector Integral Calculus

Mathematics-I — MA101

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Syllabus Coverage

Topic
Surface integrals — definition, evaluation and physical applications
Volume integrals — definition and evaluation
Green's theorem in the plane
Gauss Divergence theorem and its physical significance
Stokes' theorem and its physical significance

Chapter 1: Surface Integrals

A surface integral extends the idea of a double integral to a curved surface S in space. It is the natural tool for measuring how a scalar quantity accumulates over a surface, or how a vector field 'flows through' a surface — its flux.

1.1 Definition of a Surface Integral

Let S be a piecewise smooth surface and n the unit outward normal at a point of S. For a vector function F(x, y, z) continuous on S, the surface integral of F over S (the flux integral) is defined as:

$$\iint_S F \cdot n \, dA$$

If S is given by z = f(x, y) over a region R in the xy-plane, the surface integral can be reduced to a double integral over R by writing dA in terms of dx dy:

$$\iint_S F \cdot n \, dA = \iint_R F \cdot n \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy$$

Worked Example — Flux Through a Plane Surface

Evaluate $\iint_S F \cdot n \, dA$ where $F = z\,i + x\,j - 3y^2 z\,k$ and S is the surface $x + 2y + 2z = 6$ in the first octant, cut off by the plane $z = 0$.

Solution:

Here $f(x, y) = (6 - x - 2y)/2$, so $f_x = -1/2$, $f_y = -1$. The unit normal to the plane $x + 2y + 2z = 6$ is $n = (1/3)(i + 2j + 2k)$. Then:

$$F \cdot n = (1/3)(z + 2x - 6y^2z)$$

Since $\sqrt{(1+f_x^2+f_y^2)}\ dx\ dy = 3\ dz$... — the constant 3 from the normal cancels the Jacobian factor of 3, and the flux reduces to a straightforward double integral over the triangular base of the plane in the xy-plane; carrying this through and integrating over the triangle $0 \le x \le 6$, $0 \le y \le (6 - x)/2$ gives the value of the flux.

1.2 Physical Applications of Surface Integrals

Application	Meaning
Flux of a vector field	Net quantity of a field (fluid, electric, magnetic) crossing S per unit time
Mass of a curved lamina	$\iint_S \rho(x, y, z)\ dA$, where ρ is surface density
Centre of mass / moments of a shell	Weighted surface integrals of position over S
Electric flux (Gauss's law)	$\iint_S E \cdot n\ dA$ equals enclosed charge over ε_0

Chapter 2: Volume Integrals

2.1 Definition

Let T be a bounded closed region in space and $f(x, y, z)$ a scalar function continuous on T. Subdividing T into small volume elements ΔV_k and forming the Riemann sum $\Sigma f(x_k, y_k, z_k) \Delta V_k$, the limit as the subdivision is refined is the volume integral:

$$\iiint_T f(x, y, z)\ dV = \iiint_T f(x, y, z)\ dx\ dy\ dz$$

In cylindrical coordinates (r, θ, z), $dV = r\ dr\ d\theta\ dz$; in spherical coordinates (r, θ, φ), $dV = r^2 \sin\varphi\ dr\ d\theta\ d\varphi$. Choosing the coordinate system that matches the symmetry of T greatly simplifies evaluation.

Worked Example — Volume Integral over a Cylinder

Evaluate $\iiint_T x^2\ dV$ where T is the solid cylinder $x^2 + y^2 \le a^2$, $0 \le z \le b$.

Solution:

Using $x = r \cos\theta$, $dV = r\ dr\ d\theta\ dz$:

$$\iiint x^2\ dV = \int[z=0\to b] \int[\theta=0\to 2\pi] \int[r=0\to a]\ r^2\cos^2\theta \cdot r\ dr\ d\theta\ dz = b \cdot \pi \cdot (a^4/4) = \pi a^4 b/4$$

2.2 Physical Significance

- Total mass of a solid: $m = \iiint_T \rho(x, y, z)\ dV$, where ρ is volume density
- Total charge enclosed in a region, for a given charge density
- Centre of mass and moments of inertia of a solid body

- The right-hand side of the Gauss Divergence theorem, connecting flux to the divergence integrated over the enclosed volume

Chapter 3: Green's Theorem in the Plane

3.1 Statement

Let R be a region in the xy-plane bounded by a piecewise smooth simple closed curve C, traversed so that R lies on the left (positive/counter-clockwise orientation). If $P(x, y)$ and $Q(x, y)$ have continuous first partial derivatives in a domain containing R, then Green's theorem states:

$$\oint_C (P\,dx + Q\,dy) = \iint_R (\partial Q/\partial x - \partial P/\partial y)\,dx\,dy$$

Green's theorem converts a line integral around a closed plane curve into a double integral over the region it encloses, and is the two-dimensional counterpart of both the Divergence theorem and Stokes' theorem.

Worked Example — Verifying Green's Theorem

Evaluate $\oint_C (xy\,dx + x^2\,dy)$ where C is the boundary of the square $0 \le x \le 1$, $0 \le y \le 1$, oriented counter-clockwise.

Solution:

Here $P = xy$ and $Q = x^2$, so $\partial Q/\partial x - \partial P/\partial y = 2x - x = x$. By Green's theorem:

$$\oint_C (xy\,dx + x^2\,dy) = \int_0^1 \int_0^1 x\,dx\,dy = \int_0^1 (1/2)\,dy = 1/2$$

3.2 Applications of Green's Theorem

Application	Formula
Area enclosed by C	$A = (1/2) \oint_C (x\,dy - y\,dx)$
Verifying / replacing a line integral	Convert $\oint_C(P\,dx + Q\,dy)$ into a double integral when that is easier to evaluate
Special case of Stokes' theorem	Treat R as a flat surface in the xy-plane with unit normal k; Green's theorem is Stokes' theorem restricted to a planar field $F = P\,i + Q\,j$

Note: Taking $F(x, y, z) = P(x, y)\,i + Q(x, y)\,j + 0k$ and D a flat region in the xy-plane oriented by k, Stokes' theorem gives $\oint_C F \cdot dr = \iint_D \mathrm{curl}(F) \cdot k\,dx\,dy$, and since $\mathrm{curl}(F) \cdot k = \partial Q/\partial x - \partial P/\partial y$, this is exactly Green's theorem.

Chapter 4: Gauss Divergence Theorem

4.1 Statement (Translation Between Volume and Surface Integrals)

Let T be a closed bounded region in space whose boundary is a piecewise smooth surface S, and let $F(x, y, z)$ be a vector function with continuous first partial derivatives in a domain containing T. Then:

$$\iiint_T \mathrm{div}\,F\,dV = \iint_S F \cdot n\,dA$$

where n is the outer unit normal vector to S. Writing $F = F_1\,i + F_2\,j + F_3\,k$ and $n = (\cos\alpha, \cos\beta, \cos\gamma)$, this can also be written in component form:

$$\iiint_T (\partial F_1/\partial x + \partial F_2/\partial y + \partial F_3/\partial z)\, dx\, dy\, dz = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\, dA$$

or, expressed using projected area elements dy dz, dz dx, dx dy:

$$\iiint_T (\partial F_1/\partial x + \partial F_2/\partial y + \partial F_3/\partial z)\, dx\, dy\, dz = \iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy)$$

4.2 Physical Significance

- The divergence of a vector field at a point may be positive, negative, or zero.
- If div F > 0 at a point, the flux leaving the surface exceeds the flux entering — the point acts as a source.
- If div F < 0 at a point, the flux entering exceeds the flux leaving — the point acts as a sink.
- If div F = 0 at a point, the flux entering equals the flux leaving — the field is source-free (solenoidal) there.

Worked Example — Flux Out of a Cylinder

Evaluate $\iint_S (x^3\, dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy)$ where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$, $0 \le z \le b$, together with the circular disks z = 0 and z = b.

Solution:

Here $F_1 = x^3$, $F_2 = x^2y$, $F_3 = x^2z$, so div F = $3x^2 + x^2 + x^2 = 5x^2$. Introducing polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $dx\, dy\, dz = r\, dr\, d\theta\, dz$, and applying the Divergence theorem:

$$\iiint_T 5x^2\, dx\, dy\, dz = 5 \int_{z=0}^{b} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} r^2\cos^2\theta \cdot r\, dr\, d\theta\, dz = 5b \cdot \pi \cdot (a^4/4) = 5\pi a^4 b/4$$

4.3 Practice Problems

Evaluate $\iint_S F \cdot n\, dA$ using the Divergence theorem in each case:

- $F = [e^x, e^y, e^z]$, S the surface of the cube $-1 \le x, y, z \le 1$
- $F = [x^2, y^2, z^2]$, S the sphere $x^2 + y^2 + z^2 = 9$
- $F = [x, y, z]$, S the hemisphere $x^2 + y^2 + z^2 = 4$, $z \ge 0$, together with its base disk

Chapter 5: Stokes' Theorem

5.1 Statement

Let S be a piecewise smooth oriented surface with unit normal n, bounded by a piecewise smooth simple closed curve $C = \partial S$, positively oriented relative to n. If F is a continuously differentiable vector field on a domain containing S, then Stokes' theorem states:

$$\oint_C F \cdot dr = \iint_S (\text{curl } F) \cdot n\, dA$$

Stokes' theorem relates a line integral around C to a surface integral over any surface S having C as its boundary. Depending on convenience, either integral may be computed in terms of the other.

5.2 Applications of Stokes' Theorem

Example 1 — Computing a Line Integral via a Surface Integral

Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = y\,\mathbf{i} + xz^3\,\mathbf{j} - zy^3\,\mathbf{k}$ and C is the circle in the plane $z = -3$, $x^2 + y^2 = 4$, oriented anti-clockwise.

Solution:

To apply Stokes' theorem, choose the convenient surface S: $x^2 + y^2 \leq 4$, $z = -3$, whose boundary is C — the flat circular disc. Taking the normal $n = k$ gives S the anti-clockwise orientation matching C. Since:

$$curl(F) \cdot k = \partial/\partial x(xz^3) - \partial/\partial y(y) = z^3 - 1$$

and $z = -3$ is constant on S, $curl(F) \cdot k = (-3)^3 - 1 = -28$ throughout S. Therefore:

$$\oint_C F \cdot dr = \iint_S (curl\ F) \cdot n\ dA = \iint_S (-28)\ dA = -28 \cdot (4\pi) = -112\pi$$

Example 2 — When Stokes' Theorem Does Not Apply Directly

Consider $F(x, y) = -y/(x^2+y^2)\ i + x/(x^2+y^2)\ j$, and let C be the unit circle $x^2 + y^2 = 1$, $z = 0$, oriented anti-clockwise. A direct computation gives curl $F = 0$ wherever F is defined, and a direct evaluation of the line integral gives $\oint_C F \cdot dr = 2\pi$.

Note: This does not contradict Stokes' theorem. F is undefined at the origin (0, 0), which lies inside every disc bounded by C, so no surface S contained in the domain of F has C as its boundary — the hypotheses of Stokes' theorem simply are not met. This is the same reason F fails to be conservative even though curl $F = 0$: the domain of F is not simply connected.

5.3 Conditions for Conservativeness

Let D be a simply connected domain in $\mathbb{R}^3$ and $F: D \to \mathbb{R}^3$ a continuously differentiable vector field. The following statements are equivalent:

- F is conservative ($F = \text{grad}\ \varphi$ for some scalar potential φ).
- $\int_{C(A,B)} F \cdot dr$ is independent of the path $C(A, B)$ joining A to B.
- curl $F = 0$ throughout D.

The implication curl $F = 0 \Rightarrow$ path-independence follows from Stokes' theorem: for any simple closed curve C in D, simple connectedness guarantees an oriented piecewise smooth surface $S \subset D$ with $\partial S = C$, so $\oint_C F \cdot dr = \iint_S curl(F) \cdot n\ dA = 0$.

5.4 Physical Interpretation of Curl

Stokes' theorem gives a way of interpreting curl F in the context of fluid flow. Consider a small circular disc S_a of radius a, centred at a point P_0 in the domain of F, with unit normal n. By Stokes' theorem:

$$\oint_{C_a} (F \cdot T)\ ds = \iint_{S_a} (curl\ F) \cdot n\ dA \approx (curl\ F(P_0) \cdot n) \cdot A(S_a)$$

The quantity $F \cdot T$ is the component of F along the tangent to the boundary circle, so $\oint (F \cdot T)\ ds$ is the circulation of F around C_a. Letting $a \to 0$:

$$(curl\ F)(P_0) \cdot n = \lim_{a \to 0} (1/A(S_a)) \oint_{C_a} (F \cdot T)\ ds$$

For this reason, the normal component of curl F is called the specific circulation of the fluid at P_0, and is greatest when n is aligned with curl $F(P_0)$. If curl $F = 0$ throughout a region, there is no circulation tendency anywhere in it, and the fluid is said to be irrotational.

5.5 Practice Exercises

Using Stokes' theorem, evaluate $\oint_C F \cdot dr$ in each case:

- $F(x, y, z) = 2y\,i + 3z\,j + x\,k$, and C is the triangle with vertices A(0,0,0), B(0,2,0), C(1,1,1), oriented A → B → C → A
- $F(x, y, z) = z^2\,i + y\,j + xz\,k$, and C is the boundary of the upper hemisphere S: $z = \sqrt{4 - x^2 - y^2}$, oriented counter-clockwise
- $F(x, y, z) = x^2\,i + 4xy^2\,j + y^2x\,k$, and C is the rectangle through A(0,0,0), B(1,0,0), C(1,3,2), D(0,3,2), oriented A → B → C → D → A
- $F(x, y, z) = 3y\,i + 4x\,j + 2y\,k$, and C is the boundary of the paraboloid $z = 4 - x^2 - y^2$, $z \geq 0$, oriented counter-clockwise

Quick Reference — All Key Formulae (Module 4)

Surface Integrals

$$\iint_S F \cdot n\, dA;\ \ \text{reduces to } \iint_R F \cdot n \sqrt{1+f_x^2+f_y^2}\, dx\, dy \text{ for } z = f(x,y)$$

Volume Integrals

$$\iiint_T f\, dV;\ \ dV = r\, dr\, d\theta\, dz \text{ (cylindrical)},\ dV = r^2 \sin\varphi\, dr\, d\theta\, d\varphi \text{ (spherical)}$$

Green's Theorem

$$\oint_C (P\, dx + Q\, dy) = \iint_R (\partial Q/\partial x - \partial P/\partial y)\, dx\, dy;\ \ \text{Area} = (1/2)\oint_C (x\, dy - y\, dx)$$

Gauss Divergence Theorem

$$\iiint_T \operatorname{div} F\, dV = \iint_S F \cdot n\, dA$$

Stokes' Theorem

$$\oint_C F \cdot dr = \iint_S (\operatorname{curl} F) \cdot n\, dA$$

Conservativeness (simply connected D)

$$F \text{ conservative} \Leftrightarrow \text{path-independent} \Leftrightarrow \operatorname{curl} F = 0$$

Curl as Circulation Density

$$(\operatorname{curl} F)(P_0) \cdot n = \lim_{a \to 0} (1/A(S_a)) \oint_{C_a} (F \cdot T)\, ds$$

— End of Module 4 Notes —